Gaussian (normal) distribution

$$p(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$$

$$G(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E(G) = \langle G(x;\mu,\sigma) \rangle = \mu$$

$$Var(G) = \sigma^2 .$$

Cumulative distribution function $P(x) = \int_{-\infty}^{x} dx' \cdot p(x')$

For a normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{-t^2/2} dt = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}} \cdot \int_{-x}^{x} e^{-t^2} dt$$

Operations on normal variates

$x_1 : G(x_1, \mu_1, \sigma_1)$ $\quad$ $x_2 : G(x_1, \mu_2, \sigma_2)$

$E(x_1 + x_2) = \mu_1 + \mu_2$, $\quad Var(x_1 + x_2) = \sigma_1^2 + \sigma_2^2$

$E(x_1 - x_2) = \mu_1 - \mu_2$, $\quad Var(x_1 - x_2) = \sigma_1^2 + \sigma_2^2$

If u, v normal variables ($\mu = 0$, $\sigma = 1$)

$$Y = \frac{u}{v} \, ;$$

the distribution of Y is Cauchy:

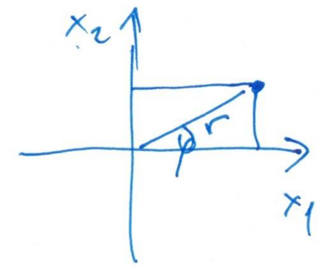
$$p(Y) = \frac{1}{\pi(1+Y^2)}$$

What is the distribution of the sum of squares of normal randoms?

$$P(x_1) = \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} \; ; \quad P(x_2) = \frac{1}{\sqrt{2\pi}} e^{-x_2^2/2} \; ; \quad dP = dx \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$y = x_1^2 + x_2^2$$

$$dP(x_1, x_2) = \frac{1}{2\pi} e^{-\frac{x_1^2 + x_2^2}{2}} dx_1 \, dx_2$$



Change to polar coordinates

$$r^2 = x_1^2 + x_2^2 = y$$

$$(x_1, x_2) \to (r, \varphi) \qquad x_1 = r\cos\varphi$$
$$x_2 = r\sin\varphi$$

$$\frac{\partial x_1}{\partial r} = \cos\varphi \qquad \frac{\partial x_2}{\partial r} = \sin\varphi$$

$$\frac{\partial x_1}{\partial \varphi} = -r\sin\varphi \qquad \frac{\partial x_2}{\partial r} = r\cos\varphi$$

$$dx_1 \, dx_2 = \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \varphi} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \varphi} \end{vmatrix} dr . d\varphi = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & +r\cos\varphi \end{vmatrix} dr d\varphi$$

$$dx_1 dx_2 = \left(r\cos^2\varphi + r\sin^2\varphi \right) dr \, d\varphi = r \, dr \, d\varphi$$

$$dP = dP(r, \varphi) = \frac{1}{2\pi} e^{-\frac{r^2}{2}} . r . dr \, d\varphi$$

Integrating over φ

$$dP(r) = dr \cdot r \cdot \frac{1}{2\pi} e^{-\frac{r^2}{2}} \cdot \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} = dr \cdot r \cdot e^{-\frac{r^2}{2}}$$

$$y = r^2$$

$$dy = 2r \cdot dr$$

$$\boxed{dP(y) = d\left(\frac{y}{2}\right) \cdot e^{-y/2}}$$

Exponential distribution.

Sum of the squares of N normal variates

$$z = \sum_{i=1}^{N} x_i^2 \qquad z \sim \chi^2_N$$

$\boxed{\text{Chi-square distribution of order } N.}$

What is the expectation value of z?

$$E(z) = \langle z \rangle = \sum_{i=1}^{N} \langle x_i^2 \rangle = \sum_{i=1}^{N} 1 = N.$$

$N=2$ was a special case $\rightarrow$ Exponential

$$p(x;N) = \begin{cases} \dfrac{x^{N/2-1} \cdot e^{-x/2}}{2^{N/2}\,\Gamma(N/2)} & \text{if } x>0 \\ 0 & \text{else} \end{cases}$$

$\Gamma(N/2)$ is the Gamma function: $\Gamma(n) = (n-1)!$ for integer n

The chi square is extremely important in statistics

Binomial

1, 2 $\quad p_1 + p_2 = 1$

1 $\quad p \quad (1-p)$

11 $\quad$ 12 $\quad$ 22

2 $\quad \frac{p \cdot p}{2!} \quad \frac{(1-p)(1-p)}{2!}$

$p \cdot (1-p)$

$(1-p) \cdot p$

$$B(k; n, p) = n! \cdot \frac{p^k}{k!} \frac{(1-p)^{n-k}}{(n-k)!}$$

$$\boxed{B = \frac{n!}{k!(n-k)!} \cdot p^k (1-p)^{n-k}}$$

$$(a+b)^n = \sum \binom{n}{k} a^k b^{n-k} \quad \binom{n}{k} \cdot p^k$$

$$\sum_k B = (p + 1 - p)^n = 1$$

Binomial distribution

Generalization of Poisson!

$$P(X=k) = \lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k}$$

$$p = \frac{\lambda}{n}$$

when $n \to \infty$, $p \to \emptyset$

$$= \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \cdot \left(1-\frac{\lambda}{n}\right)^{n-k}$$

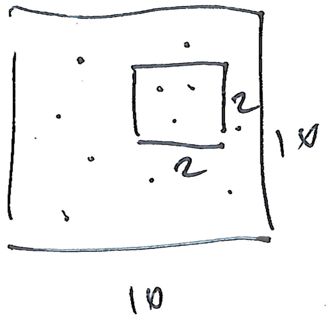
$$= \lim_{n\to\infty} \frac{n!}{(n-k)!\,k!} \cdot \left(\frac{\lambda}{n}\right)^k \cdot \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{} \cdot \underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}}_{\to 1}$$

$$= \lim_{n\to\infty} \underbrace{\left[\frac{n!}{(n-k)!} \frac{1}{n^k}\right]}_{} \left[\frac{\lambda^k}{k!} e^{-\lambda}\right] \Longrightarrow \text{Poisson}$$

$$\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-k+1}{n}\right) \to 1$$

Discrete random variables

POISSON DISTRIBUTION



λ intensity $\quad \langle n \rangle = \lambda A$

100 pts $\quad \lambda = 1 \quad 1.4 = \langle n \rangle$

$$P(n) = \boxed{\frac{(\lambda A)^n}{n!}} e^{-\lambda} \qquad \sum P(n) = 1$$

$$n \, \frac{1}{2!} \left(\underline{\frac{1}{100} \cdot A} \right)^2 = p^2$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\langle x \rangle = \sum_{n=0}^{\infty} n \frac{(\lambda A)^n}{n!} e^{-\lambda} = \sum_{n=1}^{\infty} \cancel{n} \frac{(\lambda A)^{n-1} \cdot (\lambda A)}{\cancel{n} \cdot (n-1)!} \cdot e^{-\lambda}$$

$$= (\lambda A) \underbrace{\sum_{n=1}^{\infty} \frac{(\lambda A)^{n-1}}{(n-1)!} e^{-\lambda}} = (\lambda A)$$

$$\langle n^2\rangle = \sum n^2 \frac{(\lambda A)^n}{n!} e^{-\lambda} = \sum_{n=1}^{\infty} n \cdot \frac{(\lambda A)(\lambda A)^{n-1}}{(n-1)!} e^{-\lambda} = \sum_{n=1}^{\infty} (\underbrace{n-1}+\underbrace{1}) \frac{(\lambda A)(\lambda A)^{n-1}}{(n-1)!} e^{-\lambda}$$

$$= \underbrace{\sum_{n=2}^{\infty} (n-1) \cdot \frac{(\lambda A)(\lambda A)^{n-1}}{(n-1)!} e^{-\lambda}}_{} + \underbrace{\sum_{n=1}^{\infty} (\lambda A) \cdot \frac{(\lambda A)^{n-1}}{(n-1)!} e^{-\lambda}}_{\lambda A}.$$

$$\underbrace{\sum_{n=2}^{\infty} (\lambda A)^2 \cdot \frac{(\lambda A)^{n-2}}{(n-2)!} e^{-\lambda}}_{} + \lambda A$$

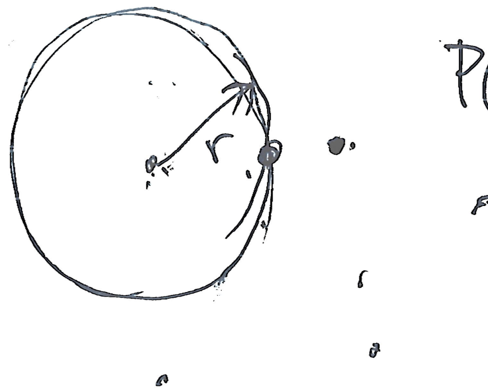
$$\langle n^2\rangle = (\lambda A)^2 + (\lambda A) = \langle n\rangle^2 + \langle n\rangle$$

$$\mathrm{Var}(n) = \langle n^2\rangle - \langle n\rangle^2 = \langle n\rangle = \sigma^2 \qquad \sigma = \sqrt{n}$$

$$I = \langle n\rangle \pm \sqrt{\langle n\rangle}$$

$$I = \langle n\rangle\left(1 \pm \frac{\sqrt{n}}{n}\right) = \langle n\rangle\left(1 + \frac{1}{\sqrt{\langle n\rangle}}\right)$$

λ $\quad A = r^2\pi$

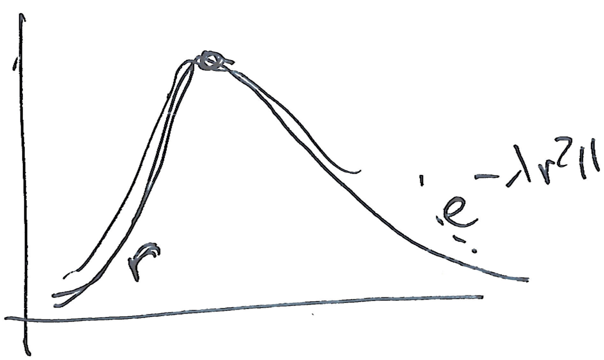


$$P(0) = \frac{(\lambda A)^0}{0!} e^{-(\lambda A)} = e^{-\lambda A}$$

$$P(0) + P(1) + P(2) \ldots\ldots (P\ldots) = 1$$

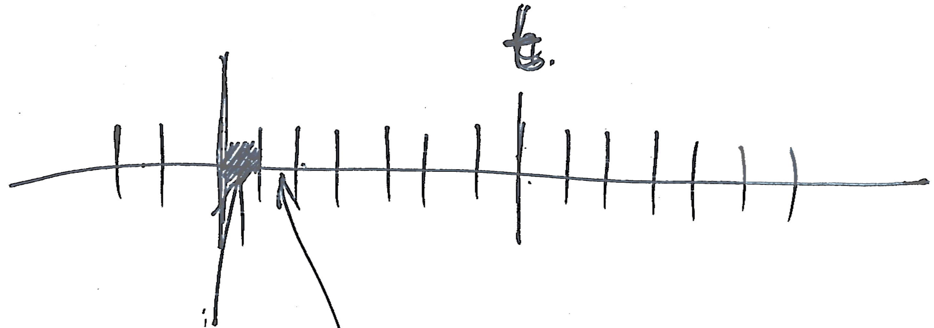
$$\boxed{P(>r) = 1 - e^{-\lambda r^2 \pi}}$$

$$\frac{dP}{dr} = P(r) = \frac{d}{dr}\left(1 - e^{-\lambda r^2 \pi}\right) = \underline{2\pi r \lambda \cdot e^{-\lambda r^2 \pi}}$$



Exponential distribution.

Arrival probabilities of Poisson



$$P(0) = e^{-\lambda} \; e^{-\lambda} \cdots\cdots e^{-\lambda} = e^{-\lambda t}$$

$1 \; \ldots\ldots \; t$

Sums of Poisson

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3$$

$$n = (n_1 + n_2 + n_3)$$

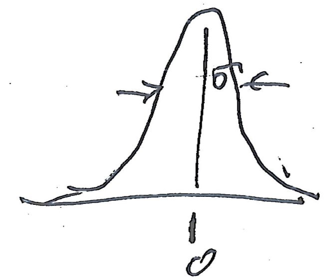
$$P_B(n_B) = \frac{\lambda_1^{n_B}}{n_B!} e^{-\lambda_1}$$

$$P_2(n_2) = \frac{\lambda_2^{n_2}}{n_2!} e^{-\lambda_2}$$

$\vdots$

Multinomial

$$\frac{n!}{\cancel{k_1!\,k_2!\cdots k_m!}}\;\frac{p_1^{k_1}}{k_1!}\,\frac{p_2^{k_2}}{k_2!}\cdots\frac{p_m^{k_m}}{k_m!} =$$

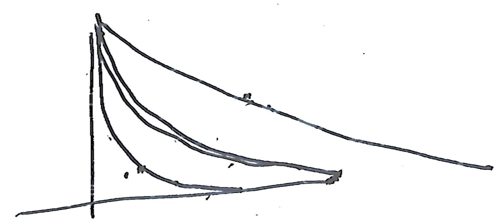


Gaussian

$$p(x) = \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \qquad \int_{-\infty}^{\infty} dx\, p(x) = 1$$

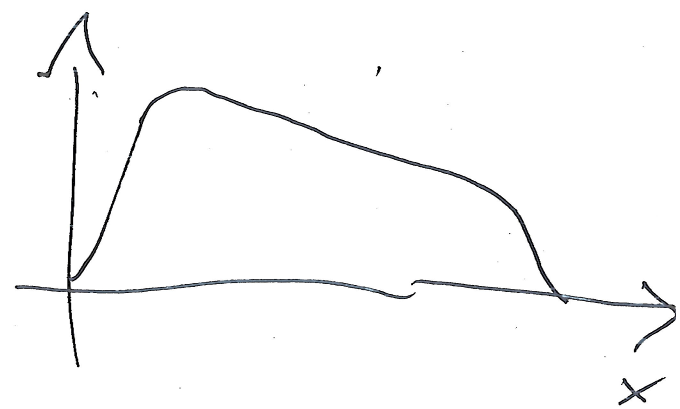
Exponential

$$p(x)\Big|_{x>0} = A\,e^{-ax}$$

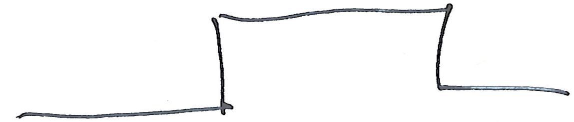


Lognormal distribution

$$\frac{e^{-\frac{(\ln x)^2}{\sigma^2}}}{\sqrt{2\pi\sigma^2}} \cdot \frac{1}{x}$$

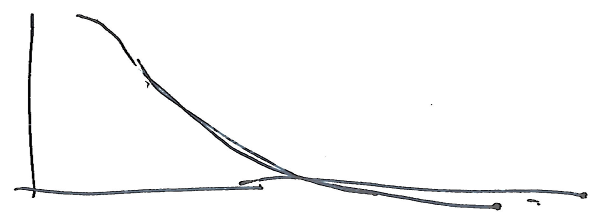


Uniform



$F(z) = 1 - e^{-\lambda z}$ $\qquad$ $P(\max(x_1 x_2) \geq z) = 1 - (1 - e^{-\lambda z})^2$

N variables $\qquad$ $P_{(1)}(\max(x_1, \dots x_N) \geq z) = 1 - \underbrace{(1 - e^{-\lambda z})}_{1-1 \to \emptyset}{}^N$



$$P_1(z) = -\frac{d}{dz}\left[1 - F(z)^N\right] = \cancel{-}N \cdot F(z)^{N-1}\left(-\frac{dF}{dz}\right)$$

$$\frac{dF}{dz} = p(z) \qquad P_{(1)}(z) = \frac{d}{dz}\left[F(z)^n\right] = \underline{\underline{N \cdot F(z)^{N-1} p(z)}}$$

$$P_{(1)}(z) = \underbrace{N \cdot F(z)^{N-1}}_{\text{all smaller}} \cdot \underbrace{p(z)}_{\text{max}}$$

ORDER STATISTIC

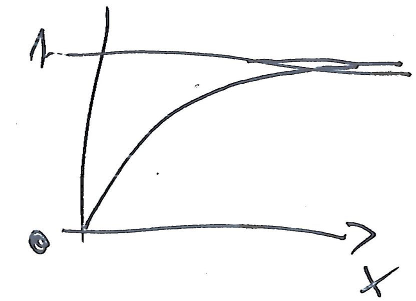
$p(x) \propto e^{-\lambda x}$

$$\int_0^\infty dx\, e^{-\lambda x} = \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^\infty = \frac{1}{\lambda}$$

Cumulative probability

$$\boxed{F(z) = P(x \le z)}$$

$$\int_0^z dx\, \cancel{\lambda} e^{-\lambda x} = \left[-\frac{\cancel{\lambda}}{\cancel{\lambda}} e^{-\lambda x}\right]_0^z = 1 - e^{\lambda z}$$



Every cumulative DF

maximum of two

What is the CDF of ~~the~~ max

(x_1, x_2) iid

M_2

$P(x_1 \le M_2) = F(M_2)$

$P(x_2 \le M_2)$.

$$P(x_1, x_2 \le M_2) = P(x_1 \le M_2)\, P(x_2 \le M_2) = F(z) \cdot F(z) = F(z)^2$$

$M_2 \to z$.

$$P_{(1)}(\cancel{x_1, x_2 \ge z}\ \max(x_1, x_2) \ge z) = \cancel{\#}\ 1 - F(z)^2$$

Extreme value distribution

$$\left[\frac{N!}{(N-2)!\,1!}\right] \cdot \underbrace{F(z)^{N-2}}_{\text{smaller}}\ \underbrace{p(z)}_{\text{2nd ranked}}\ \underbrace{(1-F(z))}_{\text{1 larger}}$$

$$\boxed{\text{EVS}}$$

$$\frac{N!}{(k-1)!\,(N-k)!} F(z)^{k-1} \cdot p(z)\,(1-F(z))^{N-k}.$$

$$P\left(\min(x_1 \dots x_N) < z\right) = (1-F(z))^N$$

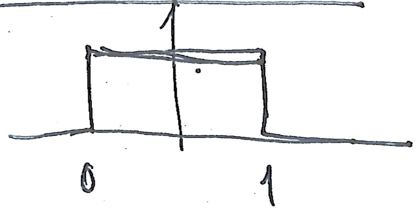
order statistics

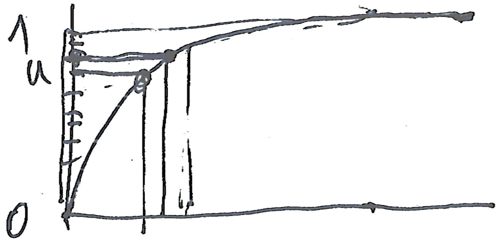
$$P(\max(x_1 \dots x_N) > z) = F(z)^N$$

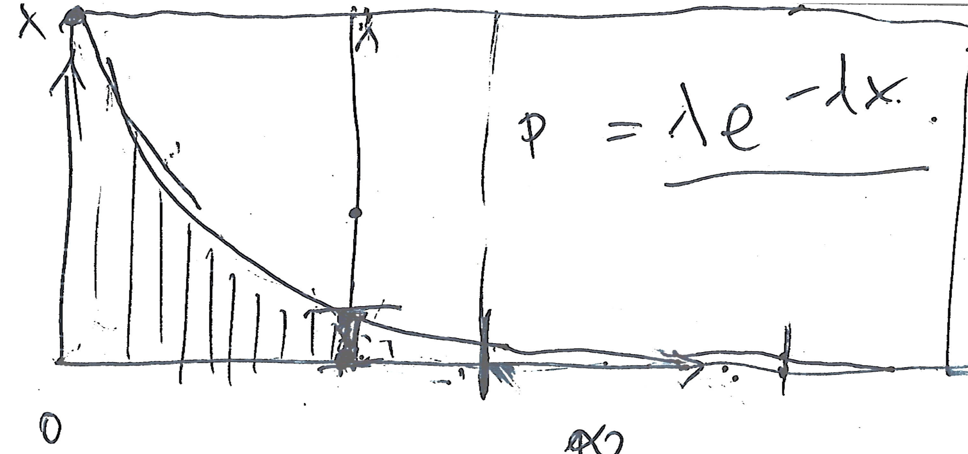
Exponential $\lambda = 1$ $\qquad p(x) = e^{-x}$

limit $\Longrightarrow$ Gumbel-distr $\quad \boxed{G(z) = e^{-e^{-z}} \rightarrow e^{-e^{-\left(\frac{z-a}{b}\right)}}}$

Generating random rand()







$x_1 = rand()$

$y_1 = \lambda . rand()$ $\lambda e^{-\lambda x}$

$F(z) = \int_0^z dx\, \lambda . e^{-\lambda x} = 1 - e^{-\lambda x}$

x_1 is good if $\underline{y_1 \le \lambda e^{-\lambda x_1}}$

$\frac{1}{\infty} = 0$

$\boxed{dF = dx . \lambda e^{-\lambda x}}$

$\frac{dF}{dx} = \lambda e^{-\lambda x}$ $du = dx \boxed{p(x)}$

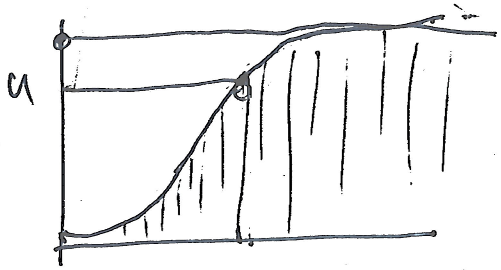
$0 \le u \le 1$

$u_1 = 1 - e^{-\lambda x_1}$

$1 - u_1 = e^{-\lambda x_1}$

$-\ln(1-u_1) = \lambda x_1$

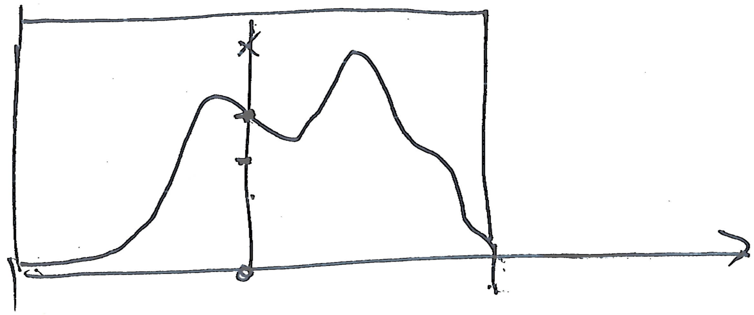
$\boxed{x_1 = -\frac{1}{\lambda}\ln(1-u_1)}$

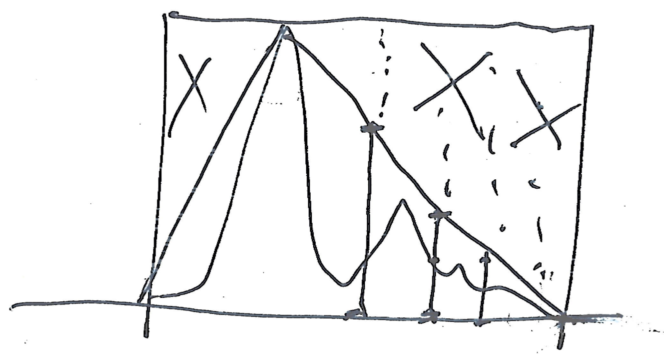


$u = F(x)$

$x = F^{-1}(u)$

Acceptance - Rejection





$g(x) \geq f(x)$

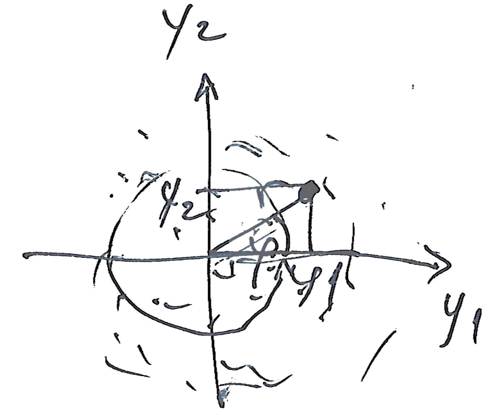
$\mathbb{R}$

$x_1, \quad y_1 = g(x_1) \cdot \mathrm{rand}()$

accept if $y_1 < f(x_1)$

Generating Gaussian

Gaussian y_1, y_2



$$dP(y_1,y_2) = e^{-\frac{y_1^2}{2}} dy_1 \cdot e^{-\frac{y_2^2}{2}} dy_2$$

$$= e^{-\frac{y_1^2+y_2^2}{2}} dy_1\, dy_2$$

$$y_1 = r \cdot \cos\varphi$$
$$y_2 = r \cdot \sin\varphi$$

cylindrical coordinate system

$$r^2 = y_1^2 + y_2^2 = r^2 (\sin^2\varphi + \cos^2\varphi) = r^2$$

$$dP(y_1,y_2) = e^{-\frac{r^2}{2}} \cdot$$

$$\frac{\partial y_1}{\partial r} = \cos\varphi \qquad \frac{\partial y_1}{\partial \varphi} = -r\sin\varphi$$

$$\frac{\partial y_2}{\partial r} = \sin\varphi \qquad \frac{\partial y_2}{\partial \varphi} = r\cos\varphi$$

$$dy_1\, dy_2 = \begin{vmatrix} \frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial \varphi} \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial \varphi} \end{vmatrix} dr\, d\varphi$$

$$J = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$$

$$dy_1\, dy_2 = r\,dr \cdot d\varphi$$

$$dP = e^{-\frac{r^2}{2}} \cdot \underbrace{r \cdot dr}_{d\left(\frac{r^2}{2}\right)} \cdot d\varphi = \underline{e^{-u}\, du} \cdot \underline{d\varphi}$$

$$u = r^2/2$$

$$\boxed{\varphi_1 = 2\pi\ \mathrm{rand}()}$$

$$\boxed{u_1 = -\ln(1-\mathrm{rand}())} \Rightarrow r = \sqrt{2u}$$

$$y_1 = r \cdot \cos\varphi$$

$$y_2 = r\sin\varphi$$