FOURIER TRANSFORMS II.

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-2\pi i kx} f(x)\,dx \qquad (1)$$

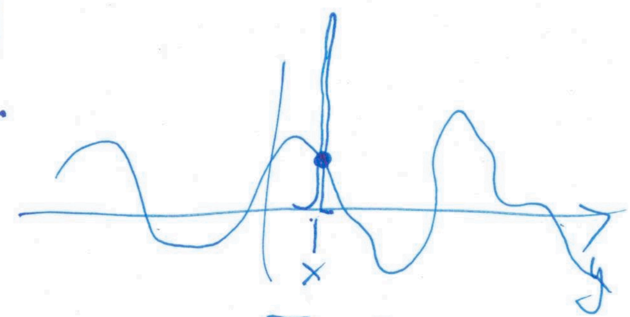
Inverse transform
$$f(x) = \int_{-\infty}^{\infty} dk.\, e^{2\pi i kx}\, \hat{f}(k) \qquad (2)$$

$$f(x) = \int_{-\infty}^{\infty} dk.\, e^{2\pi i kx}\, \underline{\hat{f}(k)} = \int_{-\infty}^{\infty} dk.\, e^{2\pi i kx} \cdot \int_{-\infty}^{\infty} e^{-2\pi i ky} \cdot f(y)\,dy$$

$$f(x) = \int_{-\infty}^{\infty} dy \cdot f(y) \cdot \underbrace{\int_{-\infty}^{\infty} dk.\, e^{2\pi i kx}\, e^{-2\pi i ky}} = \underline{\int_{-\infty}^{\infty} dy \cdot f(y)\, \delta(x-y)}$$

$$\boxed{\begin{array}{c} \underline{\displaystyle\int_{-\infty}^{\infty} dk.\, e^{2\pi i k(x-y)}} \\ \underline{\delta(x-y)} \end{array}}$$

Dirac delta

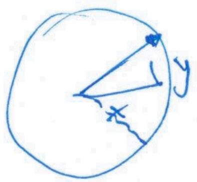


Dual representation
time-domain vs. frequency domain

$$\hat{f}(k) = \int_{-\infty}^{\infty} dx \, e^{-2\pi i k x} f(x) \quad = x(k) + i\, y(k) = a(k)\, e^{i\varphi(k)}$$

complex number $z = x + iy = a e^{i\varphi}$

Re Im amplitude phase



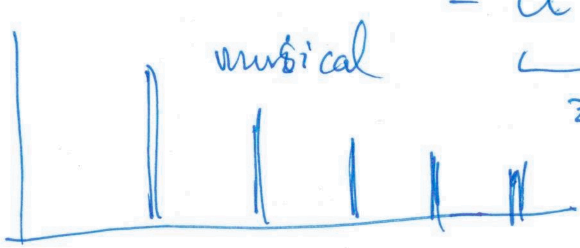
Frequency spectrum

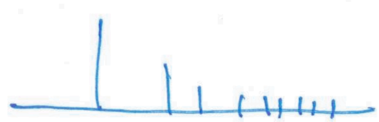
Power spectrum.

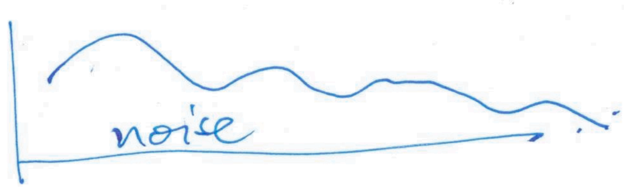
$$P(k) = |\hat{f}(k)|^2 = x(k)^2 + y(k)^2 = a^2$$

$$|z|^2 = z \cdot z^* = (x+iy)(x-iy) = x^2+y^2$$

$$= \underbrace{a e^{i\varphi}}_{z} \cdot \underbrace{a \cdot e^{-i\varphi}}_{z^*} = a^2$$

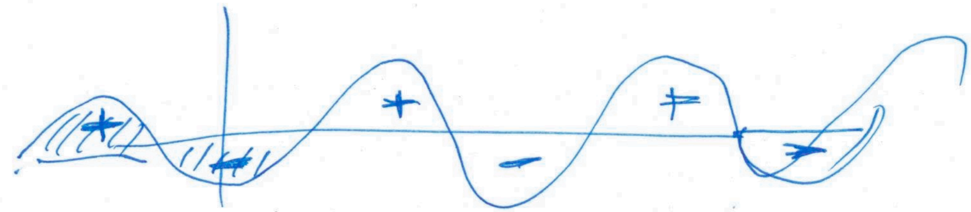






AC DC $\equiv$ zero freq

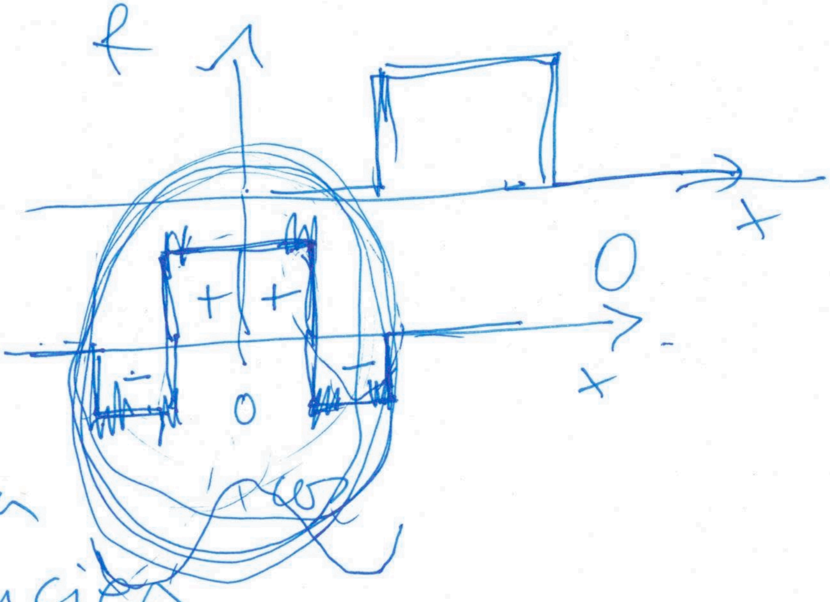
$$\int_{-\infty}^{\infty} f(x)\,dx = \cancel{0} = \mu$$

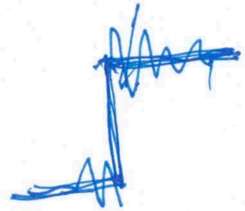


$\sin 0.x = 0$

$\cos 0 x = 1$

$$\int_{-\infty}^{\infty} |f(x)|^2 dx$$





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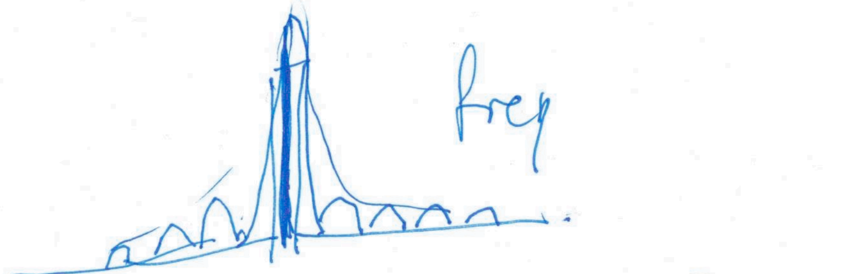
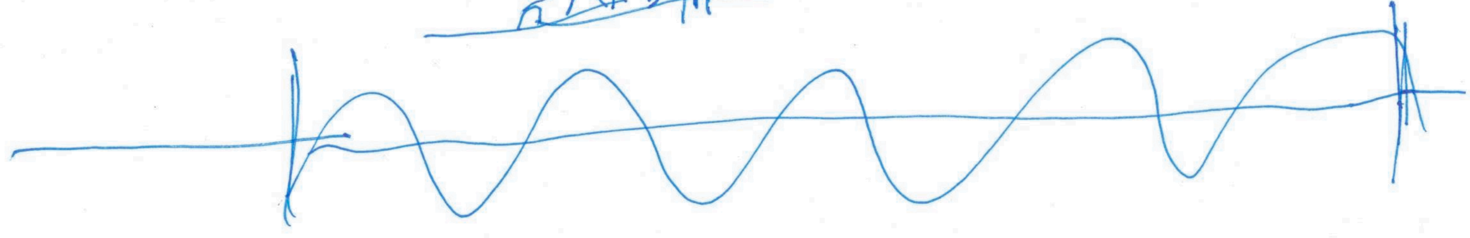
"GIBBS" phenomenon

Infinitely high frequencies

Conjugate relation





~~For~~ Sine wave infinitely sharp in k-space but totally broad in x-space

Vica versa, a Dirac delta is sharp in x-space, but very broad in k-space.

These are extremes. Consider a ~~sine wave~~ signal with finite energy, but zero mean.

$$\int_{-\infty}^{\infty} f(x)\,dx = 0 \quad \Rightarrow \mu$$

$$a^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \qquad e$$

We can normalize with a, so assume that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 1 \qquad \text{like a probability density}$$

$$\hat{f}(k) = \int_{-\infty}^{\infty} dx \cdot e^{-2\pi i k x} f(x).$$

What is the integral of $|\hat{f}(k)|^2$?

$$\int_{-\infty}^{\infty} dk \cdot |\hat{f}(k)|^2 = \int_{-\infty}^{\infty} dk \cdot \int_{-\infty}^{\infty} dx\, e^{-2\pi i k x} f(x) \cdot \int_{-\infty}^{\infty} dy \cdot e^{2\pi i k y} f(y) =$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, f(x) f(y) \underbrace{\int_{-\infty}^{\infty} dk \cdot e^{-2\pi i k (x-y)}}_{\delta(x-y)} = \int_{-\infty}^{\infty} dx \cdot |f(x)|^2$$

Parseval's theorem.

Fourier transform ~~has an inverse~~:

Fourier transform

$$\hat{f}_1(k) = \int_{-\infty}^{\infty} e^{-2\pi i k x} f(x)\, dx$$

Power spectrum

$$P(k) = |\hat{f}_1(k)|^2$$

Change of origin

$$\hat{f}_2(k) = \int_{-\infty}^{\infty} e^{-2\pi i k x} f(x-x_0)\, dx =$$

$$= \int e^{-2\pi i k (x-x_0)} e^{-2\pi i k x_0} \cdot f(x-x_0) dx =$$

$$= e^{-2\pi i k x_0} \hat{f}_1(k) .$$

We have an inverse transform:

$$f(x) = \int_{-\infty}^{\infty} dk\, e^{2\pi i k x} \cdot \hat{f}_1(k) =$$

$$= \int_{-\infty}^{\infty} dk \cdot \cancel{f_1} e^{2\pi i k x} \int_{-\infty}^{\infty} dy \cdot f(y)\, e^{-2\pi i k y} =$$

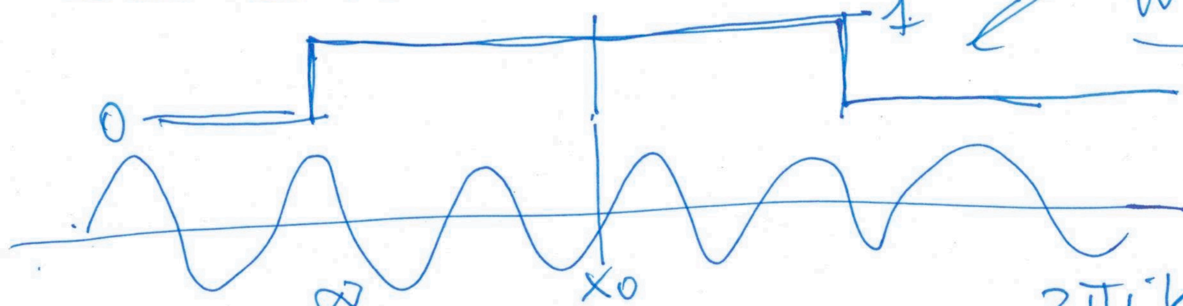
$$= \int_{-\infty}^{\infty} dy\, f(y) \cdot \underbrace{\int_{-\infty}^{\infty} dk \cdot e^{-2\pi i (y-x)}}_{\delta(y-x)}$$

Dirac delt

$$\int_{-\infty}^{\infty} dy\, f(y)\, \delta(y-x) = f(x)$$

Convolution theorem

Window function



$$\hat{f}_W(k) = \int_{-\infty}^{\infty} \underbrace{f(x)\cdot W(x-x_0)}\, \cancel{dx}\, e^{-2\pi i k x}\, dx \qquad (3)$$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k'')\cdot e^{2\pi i k'' \cancel{x}}\, dk'' \qquad (4)$$

$$W(x-x_0) = \int_{-\infty}^{\infty} \hat{W}(k')\, e^{2\pi i k'\cdot(x-x_0)} \cdot dk' \qquad (5)$$

(4)(5) → (3)

$$\hat{f}_W(k) = \int_{-\infty}^{\infty} dx\, e^{-2\pi i k x} \cdot \int_{-\infty}^{\infty} dk''\, \hat{f}(k'') \cdot e^{2\pi i k'' x} \cdot \int_{-\infty}^{\infty} dk'\, \hat{W}(k')\, e^{2\pi i k'(x-x_0)}$$

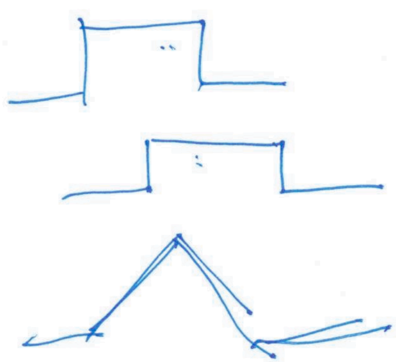
$$\hat{f}_W(k) = \left[\int_{-\infty}^{\infty} dk'\, dk''\cdot e^{-2\pi i k' x_0}\, \underbrace{W(k')}\, \hat{f}(k'') \underbrace{\int_{-\infty}^{\infty} dx \cdot e^{-2\pi i x \underbrace{(k-k''-k')}}}_{\delta(k-k''-k')}\right]$$

$$f_w(k) = \int_{-\infty}^{\infty} dk'\, dk'' \cdot \hat{f}(k'')\, \hat{w}(k')\, e^{-2\pi k' x_0} \cdot \delta(k - k' - k'')$$

$$= \int_{-\infty}^{\infty} dk'\, \hat{w}(k')\, e^{-2\pi k' x_0} \cdot \underbrace{\int_{-\infty}^{\infty} dk'' \cdot f(k'') \cdot \delta(k - k' - k'')}_{f(k-k')}$$

$$= \int_{-\infty}^{\infty} dk'\, w(k')\, f(k - k') \cdot \underbrace{\cancel{e^{-2\pi k' x_0}}}_{1}$$

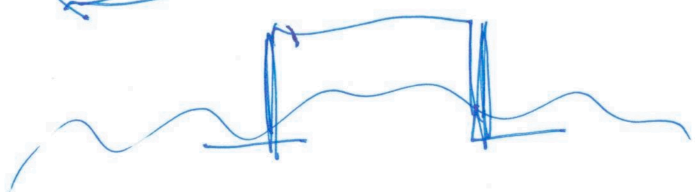
Convolution

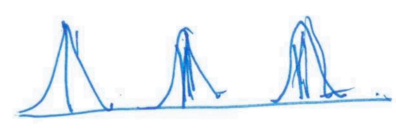


$f(k) =$ 

$k - k'$

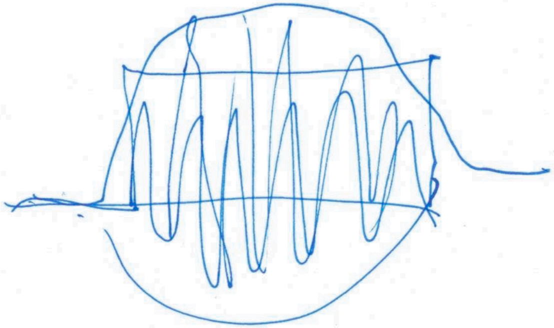
$w(k) =$ 

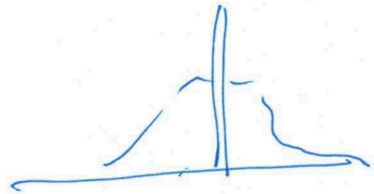


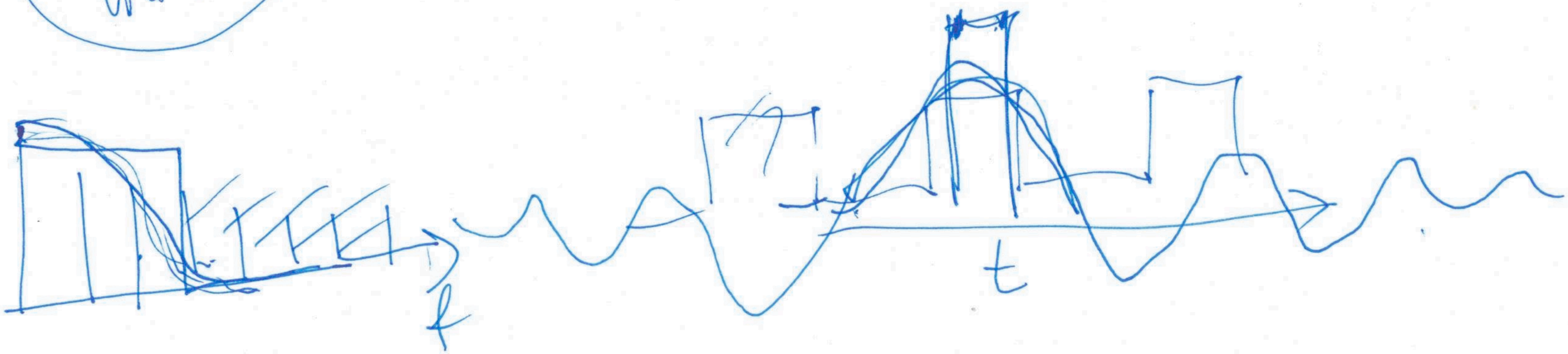


Convolution → makes signals broader.

windowing → makes signals narrower



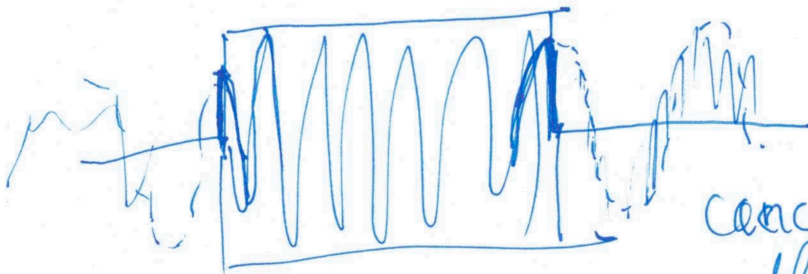




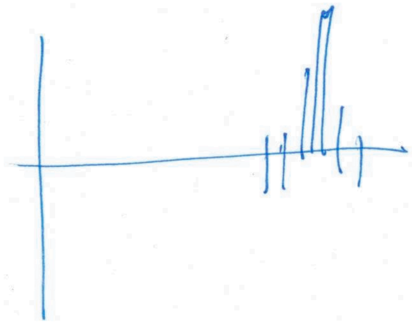
Low-pass filter

convolution ⟺ window

× ⟺ f

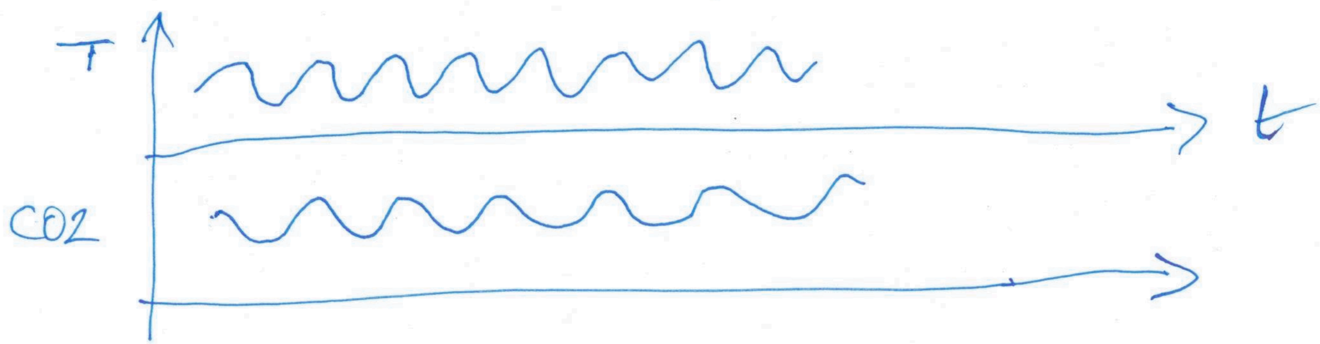


cancel all outside

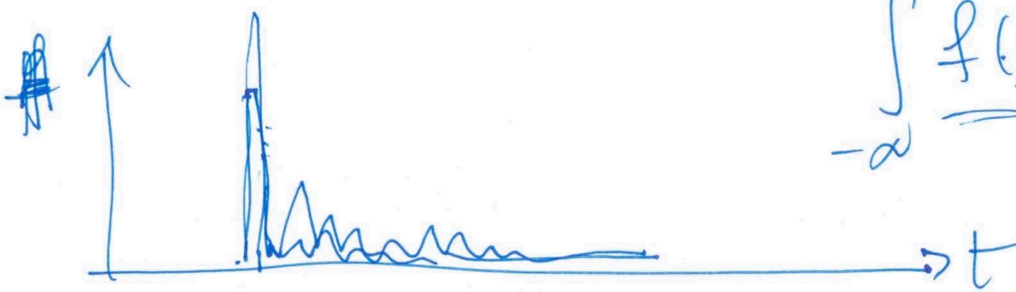


Convolution $\int_{-\infty}^{\infty} f(x)\,g(y-x)\,dx = f * g$

Correlation $\int_{-\infty}^{\infty} f(x)\,g(x-y)\,dx = C(f,g)$

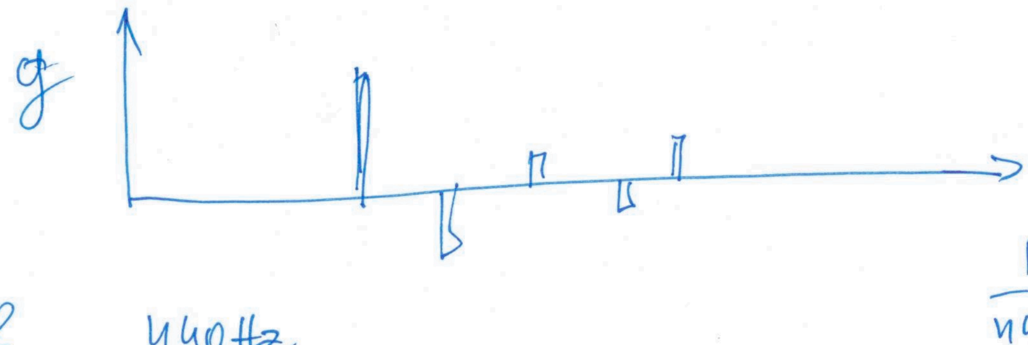


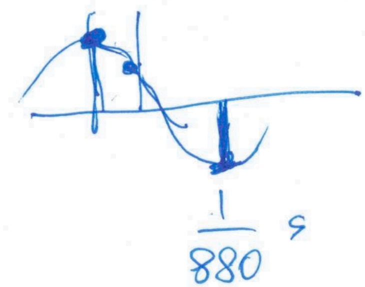
Autocorrelation



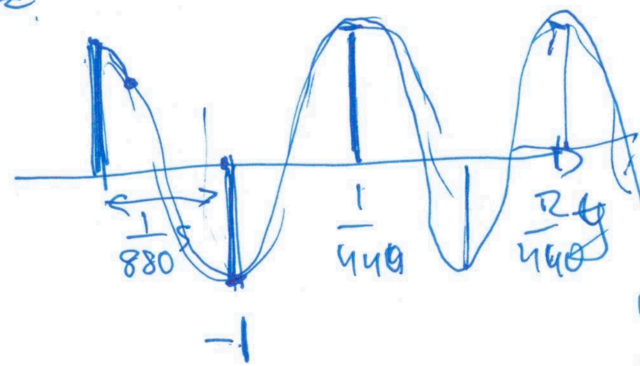
$$\int_{-\infty}^{\infty} f(x)\, f(x-y)\, dx =$$

$$\int_{-\infty}^{\infty} f(x)\, f(x-y)\, dx$$





f 440Hz.



$$|\hat{f}(k)|^2 \geq 0$$

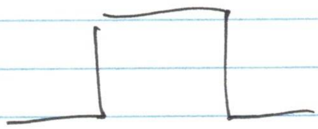
$$c(y) = \int_{-\infty}^{\infty} f(x)\, f(x-y)\, dx = \int_{-\infty}^{\infty} dk\, \underline{P(k)} \cdot e^{2\pi i k y}$$

power spectrum

correlation positive definite function | Fourier transform pair

Convolution theorem

multiply in x $\leftrightarrow$ convolve in k.



effect of a finite window

$\rightarrow$ broadening of lines.

Effect of a convolution: smoothing.

low pass filtering

Correlations

$\varphi(r) = \int f(x) f(x+r)$ auto correlation

~~cross correlation~~.

Fourier transform of the power spectrum

Cost of FT $\rightarrow$ FFT

How can we characterize the width?

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx \cdot x^2 \cdot |f(x)|^2 \qquad \langle k^2 \rangle = \int_{-\infty}^{\infty} dk \cdot k^2 \cdot |\hat{f}(k)|^2$$

One can show

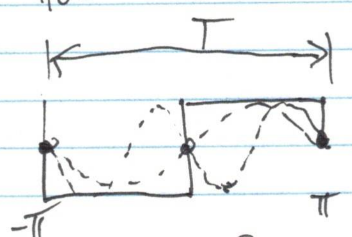
$$\boxed{\langle x^2 \rangle^{1/2} \langle k^2 \rangle^{1/2} \geq \frac{1}{4\pi}}$$

Proof is in the notes.

~~Wavelets~~: This makes Fourier transforms less than optimal for representing sharp objects..

Gibbs - phenomenon.

How do ~~we~~ expand a square wave?

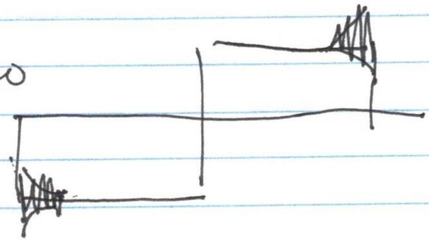


keep adding a finite term

$$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ +1 & \text{for } x \geq 0 \\ 0 & x = 0, \pm\pi \end{cases}$$

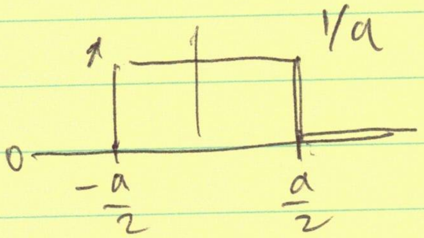
Expansion is $f(x) = \frac{1}{2} + \sum_{n=0}^{\infty} a_n \cdot \sin(nx)$.

~~Here~~



Windowing

sinc function



$$w(x) = \begin{cases} \frac{1}{a} & |x| < \frac{a}{2} \\ 0 & \text{else} \end{cases}$$

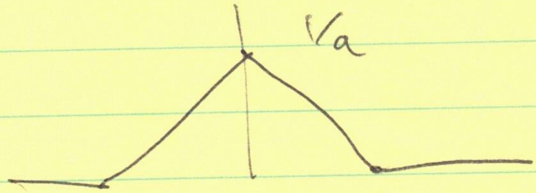
$$\hat{w}(k) = \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-2\pi i k x} \, dx = \frac{1}{a}\left[\frac{-1}{2\pi i k} e^{-2\pi i k x}\right]_{-\frac{a}{2}}^{a/2}$$

$$= -\frac{1}{2\pi i k a}\left[e^{-2\pi i k a/2} - e^{+2\pi i k a/2}\right] =$$

$$= \frac{1}{\pi k a}\left[\frac{e^{+i\pi k a} - e^{-i\pi k a}}{2i}\right] =$$

$$= \frac{\sin \pi k a}{\pi k a} = j_0(\pi k a)$$

What is the window function of



Fourier transform, central limit theorem

→ Fourier amplitudes are Gaussian

- DFT
- FFT
- Short time FFT, periodogram

$=$

$f(x)$ $\quad \int dx\,[f(x)^2] = \sigma^2$ duality

$F(k)$ $\quad \int dk \cdot [F(k)]^2 = \sigma^2$ variance is the same

Heisenbergs inequality

$\int |f(x)|^2 dx = 1 \qquad \int |\hat{f}(s)|^2 ds = 1$

$|f(x)|^2$ like a probability density.

Convolution

$$\cancel{\#}(\hat{f}(x)) \cdot \cancel{\#}(\hat{g}(x)) = \hat{h}(k)$$

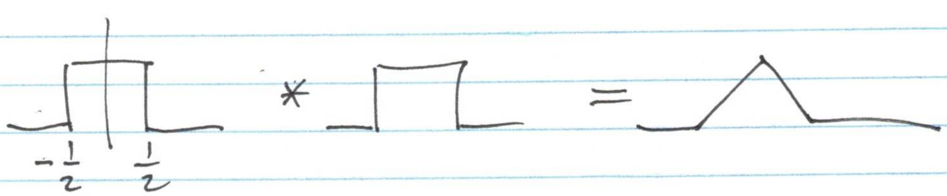
what is $h(x)$.

$$\hat{h}(k) = \int d\cancel{x} \cdot e^{-2\pi i k x} f(x) \int dy \cdot e^{-2\pi i k y} g(y) =$$

$$= \int_{-\infty}^{\infty} dx \cdot f(x) \int_{-\infty}^{\infty} dy\, g(y)\, e^{-2\pi i k \cdot (x+y)}$$

$$= \int dx\, f(x) \cdot \int_{-\infty}^{\infty} du \cdot g(u-x)\, e^{-2\pi i k u}$$

$$= \int du \cdot \underbrace{\left[\int dx\, f(x)\, g(u-x)\right]}_{h(x)} e^{-2\pi i k u}$$

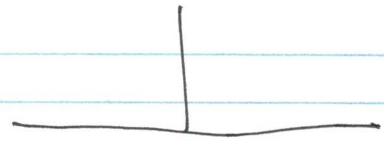


$$f(x) = \begin{cases} 1 & \text{if } |x| \le \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

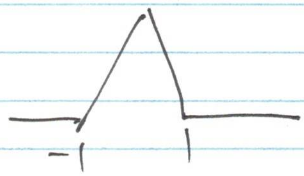
$$\hat{f}(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \cdot e^{-2\pi i k x} = \cancel{\#}' \int_0^{\frac{1}{2}} dx \left[e^{-2\pi i k x} + e^{2\pi i k x}\right] =$$

$$= 2 \int_0^{\frac{1}{2}} dx \cdot \cos 2\pi k x = 2 \cdot \int_0^{\frac{1}{2} \cdot 2\pi k} d(2\pi k x) \cos 2\pi k x =$$

$$\frac{2}{2\pi k}\int_0^{\pi k} du \cdot \cos u =$$



$$= \frac{2}{2\pi k}\cdot\left[\sin \pi k - \sin 0\right] = \left(\frac{\sin \pi k}{\pi k}\right) = \mathrm{sinc}(\pi k)$$



$$\rightarrow \left(\frac{\sin \pi k}{\pi k}\right)^2$$