Convolution

$$\cancel{\#}\left(\hat{f}(x)\right) \cdot \cancel{\#}\left(\hat{g}(x)\right) = \hat{h}(k)$$

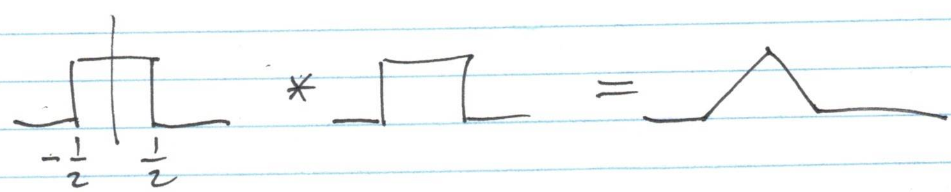
What is $h(x)$.

$$\hat{h}(k) = \int dx \cdot e^{-2\pi i k x} f(x) \int dy \cdot e^{-2\pi i k y} g(y) =$$

$$= \int_{-\infty}^{\infty} dx \cdot f(x) \int_{-\infty}^{\infty} dy\, g(y)\, e^{-2\pi i k \cdot (x+y)}$$

$$= \int dx\, f(x) \cdot \int_{-\infty}^{\infty} du \cdot g(u-x)\, e^{-2\pi i k u}$$

$$= \int du \cdot \underbrace{\left[\int dx\, f(x)\, g(u-x)\right]}_{h(x)} e^{-2\pi i k u}$$

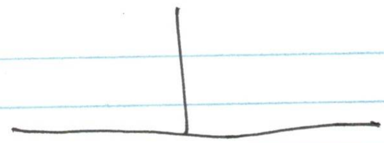


$$f(x) = \begin{cases} 1 & \text{if } |x| \le \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

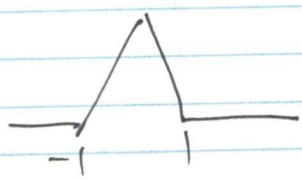
$$\hat{f}(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \cdot e^{-2\pi i k x} = \int_0^{\frac{1}{2}} dx \left[e^{-2\pi i k x} + e^{2\pi i k x} \right] =$$

$$= 2 \int_0^{\frac{1}{2}} dx \cdot \cos 2\pi k x = 2 \cdot \int_0^{\frac{1}{2} \cdot 2\pi k} d(2\pi k x) \cos 2\pi k x =$$

$$\frac{2}{2\pi k}\int_0^{\pi k} du \cdot \cos u =$$



$$= \frac{2}{2\pi k}\cdot\left[\sin \pi k - \sin 0\right] = \left(\frac{\sin \pi k}{\pi k}\right) = \cancel{\mathrm{sinc}}(\pi k)$$



$$\rightarrow \left(\frac{\sin \pi k}{\pi k}\right)^2$$

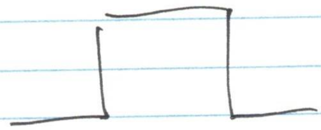
[Correlation of a random sequence with itself is * Dirac delta

[Fourier transform is uniform 1.

"white noise"

Convolution theorem

multiply in x $\leftrightarrow$ convolve in k.



effect of a finite window

$\rightarrow$ broadening of lines.

Effect of a convolution: smoothing.

low pass filtering

Correlations

$c(r) = \int f(x) f(x+r)$ autocorrelation

~~cross correlation~~.

Fourier transform of the power spectrum

Cost of FT $\rightarrow$ FFT

$$c(r) = \int_{-\infty}^{\infty} f(x)\, f(x-r)\,dx = \int_{-\infty}^{\infty} P(k)\, e^{2\pi i k r}\, dr$$

Discrete Fourier transform

Sampling at f_s, for T seconds.

The number of samples is $N = T . f_s$.

Fourier transform for x_n $\quad n = 0, \dots N-1$

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-2\pi i k (n/N)}$$

Mirror (negative frequencies)..

$$X_{N-k} = \sum_{n=0}^{N-1} x_n \cdot e^{-2\pi i n (N-k)/N}$$

$$e^{-2\pi i \frac{n}{N} \cdot (N-k)} = \underbrace{e^{-2\pi i n}}_{=1} \cdot e^{2\pi i \frac{n}{N} k} \qquad k \to -k$$

$$\boxed{X_{N-k} = X_{-k} = X_k^*}$$

Count the free parameters:

Real has N parameters

N even . $X_{N-k} = X_k^*$

N odd $\quad X_0 = X_0$

$X_{N-k} = X_k^*$

$$X(t) = A . e^{2\pi i f . t} \qquad t = n.(\Delta t)$$

$$X_n = A . e^{2\pi i f . n . (\Delta t)} .$$

What is Δt? Time between two samples:

$$\Delta t = \frac{1}{f_s}$$

The whole signal has N samples;

$$N = T . f_s$$

$$X_k = \sum x_n \cdot e^{-2\pi i \frac{kn}{N}} *$$

$$\frac{kn}{N} = \left(\frac{k}{\Delta T}\right)(n \Delta t) . \frac{1}{N} = (k f_0) . (n \Delta T)$$

$$f_0 = \frac{1}{N . \Delta t} = \left(\frac{f_s}{N}\right) = \frac{1}{T}$$

$f_s = 40{,}000\,Hz$, $T = 5$ sec . $f = 600\,Hz$.

what would be k at the peak?

$$N = 5 . \times 40000 = 200000$$

$$f_0 = 0.2\,Hz . \qquad k = \frac{600}{0.2} = 3000$$

Discrete sampling

Sample is measured at uniformly spaced intervals.

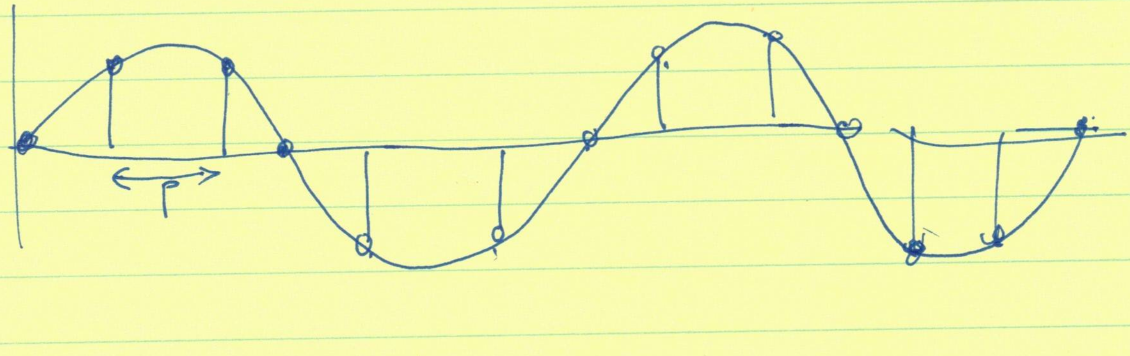
We have a continous signal $f(x)$.
We sample at p intervals

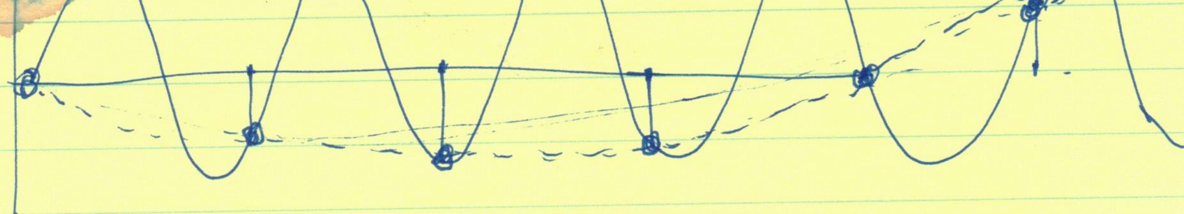
$$f'(x) = \sum_{k=\infty}^{\infty} f(x-p.k) = \sum_{k=-\infty}^{\infty} f(x).\delta(x-kp)$$

$$= f(x) \underbrace{\sum_{k=-\infty}^{\infty} \delta(x-kp)}$$

$Ш_p * f$ 'shah'. function

Sampling a sine function





Nyquist frequency

sampling rate $> \underline{\underline{2\nu_{max}}}$

Band-limited function.

This is why CD's are sampled at 44.1 kHz.

Mirror frequencies

When we sample a function at f_s frequency

$\Delta t = \left(\frac{1}{f_s}\right)$

$\sin(2\pi(f + Nf_s)t + \phi) \quad N = 0, \pm 1, \ldots \pm 2$

yield identical samples

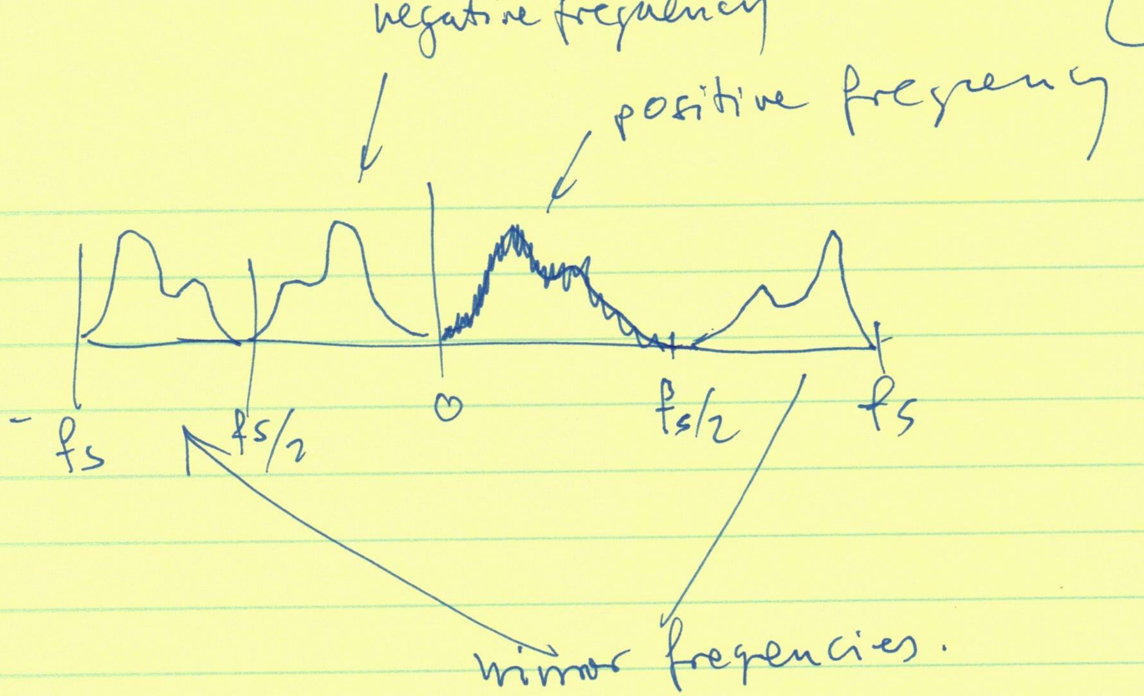
$\sin\left(2\pi(f+Nf_s)k\frac{1}{f_s} + \phi\right) =$

$= \sin\left(2\pi\left(f\frac{k}{f_s}\right) + 2\pi Nk + \phi\right)$

e.g.: $\quad f = 0.1 f_s \qquad N = -1$

$f \cdot 0.1 f_s - f_s = -0.9 f_s \equiv 0.9 f_s$

Mirror freq.



We have a 10kHz

Nonlinearitie, distortion

$f \quad f^2 \quad f^3$

10 20. 30. 40

$$\begin{array}{r} 44.1 \\ \underline{30.0} \\ \underline{14.1\ \text{kHz}} \end{array} \qquad \begin{array}{r} 44.1 \\ \underline{40.0} \\ \underline{4.1\ \text{kHz}} \end{array}$$

Aliasing

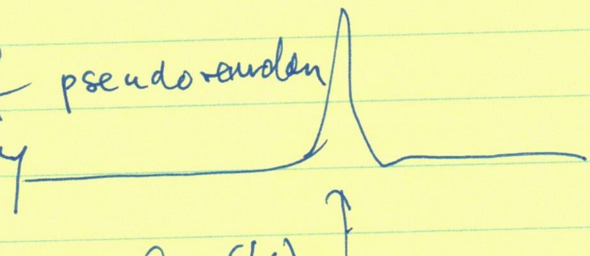
Keep doing this ~~aa~~ for $\log_2 N$ steps
and we have a single element.
The total cost of the 1D FFT is $\propto N \log N$.
instead of N^2.

Differences in the prefactor, typically ~ 6.

minimize K FFTW fastes FFT of the world.

Cell phone GPS.

Cross correlation of pseudorandom
peak gives the time delay



If we know how many peaks (K)
the cost can be as small as

$K . \log N$. Ingremental FFT.

Inverse FFT.

FFT shift

Fast Fourier Transform (FFT) → FFTW

N samples $$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-2\pi i k (n/N)}$$ $\boxed{\text{DFT}}$

t

This requires $O(N^2)$ operations (complex multiplications)

Imagine that N is a power of 2.

Cooley-Tukey

$$X_k = \underbrace{\sum_{m=0}^{N/2-1} x_{2m}\, e^{-2\pi i (2m)k/N}}_{E_k} + \underbrace{\sum_{m=0}^{N/2-1} x_{2m+1}\, e^{-\frac{2\pi i (2m+1)k}{N}}}_{\boxed{e^{-\frac{2\pi i k}{N}}} \cdot O_k}$$

DFT is periodic: $\left[\begin{array}{l} E_{k+\frac{N}{2}} = E_k \\ O_{k+\frac{N}{2}} = O_k \end{array}\right]$, period for the whole is N

$$X_{k+N} = X_k$$

~~For a period~~

$$X_k = \begin{cases} E_k + e^{-\frac{2\pi i}{N}k} \cdot O_k & 0 < k < \frac{N}{2} \\ E_{k-N/2} + e^{-\frac{2\pi i k}{N}} O_{k-\frac{N}{2}} & \frac{N}{2} < k \leq N \end{cases}$$

$$e^{-\frac{2\pi i}{N}\left(k+\frac{N}{2}\right)} = e^{-\frac{2\pi i}{N}k} \cdot \underbrace{e^{-\frac{2\pi i}{N}\cdot\frac{N}{2}}}_{e^{-i\pi} = -1}.$$

$$X_k = E_k + e^{-\frac{2\pi i}{N}k} O_k$$

$$X_{k+\frac{N}{2}} = E_k - e^{-\frac{2\pi i}{N}} O_k$$

half the computation.

computational complexity

$C(N)$ is the cost of the FFT for N points

$$C(N) = 2\boxed{C(N/2)} + \underline{KN}$$

assembly cost for sub arrays

$n = \log_2 N$

$$T(n) = \frac{C(N)}{N} = \frac{C(N/2)}{N/2} + K$$

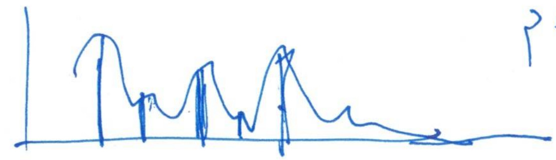
$$T(n) = T(n-1) + K$$

$$T(n) = nK$$

$$\boxed{C(N) = KN \cdot \log N}$$

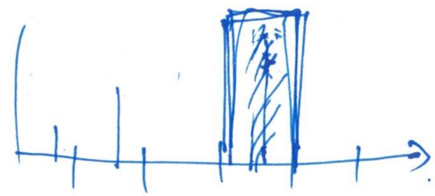
Sparse FFT   Indyk   randomized algorithms.

$$\begin{matrix} \mathrm{FFT} \\ \mathrm{FFT} \end{matrix} \rightarrow \otimes \rightarrow \mathrm{FFT}^{-1} \longrightarrow$$



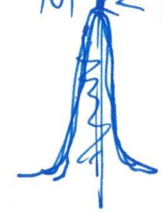
perfect.

Prior knowledge   Pseudo-random $\Longrightarrow$ 1 peak in x coor



N/K

$M << N$



$$N^2 \rightarrow \left(\frac{N}{K}\right) M << N^2$$

$$\boxed{M \log N}$$